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The complete field of charged perfect fluid spheres and of other static spherically symmetric charged distributions

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Abstract. Exact solutions are presented which describe the complete field (in the interior and exterior) of the following static spherically symmetric systems: (I) charged perfect fluid spheres and (II) charged distributions with no radial stress.

Perfect fluidity in systems (I) and radial stress in systems (II) are defined in two distinct ways and exact solutions are obtained in each case. All the solutions are well behaved everywhere and satisfy the necessary boundary conditions across the surface of the distribution as well as the elementary flatness conditions at the centre of the distribution. Outside the distribution, the solutions are identical to the (exterior) Nordström solution. Inside the distribution the solutions for systems (I) reduce to the interior Schwarzschild solution on setting the charge density equal to zero. The solutions for systems (II) reduce, on setting the charge density to zero, to the 'new interior Schwarzschild solution' obtained by the author (1974). It is suggested that the solutions for systems (II) may represent the field inside a *charged* Einstein cluster.

1. Introduction

In a recent paper (Florides 1977), to be referred to as paper A, the author derived the general solution, representing the complete field of the most general static spherically symmetric charged distribution, of the combined Einstein–Maxwell field equations

$$(\sqrt{-g}F^{ij})_{,j} = \sqrt{-g}J^i, \qquad F_{[ij,k]} = 0, \qquad G^i_j + \kappa T^i_j = 0, \qquad (\kappa = 8\pi). \tag{1.1}$$

Here T^i_j stands for

$$T^i_j = M^i_j + E^i_j, \tag{1.2}$$

where M^i_j is the matter energy tensor and E^i_j, given by

$$4\pi E^i_j = F^{ia}F_{ja} - \tfrac{1}{4}\delta^i_j F^{ab}F_{ab}, \tag{1.3}$$

is the electromagnetic energy tensor; J^i is the 4-current density vector given by

$$J^i = 4\pi\sigma V^i,$$

where V^i, with $V_iV^i = -1$, is the 4-velocity, and σ is the proper charge density, of the charge. The rest of the notation† is standard.

† Relativistic units are used so that the speed of light in vacuum and the gravitational constant are both unity, and the signature of the metric is +2.

The history of the static, spherically symmetric, distribution of charge is a time-like world-tube whose cross-section x^4 = constant is bounded by a spherical surface S. In terms of curvature coordinates $x^i = (r, \vartheta, \varphi, t)$, with spatial origin at the centre of S, the equation of S is

$$r = a. \tag{1.4}$$

It was shown in paper A that the complete field of the distribution is given by the line element

$$ds^2 = e^{\alpha(r)}\, dr^2 + r^2\, d\Omega^2 - e^{\gamma(r)}\, dt^2, \tag{1.5}$$

with

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2, \tag{1.6}$$

where the functions $\alpha(r)$ and $\gamma(r)$ are the solutions of (1.1) to (1.3); they are given, for all values of r, by the equivalent expressions

$$e^{-\alpha(r)} = 1 - \frac{\kappa}{r}\int_0^r \left(\rho + \frac{1}{\kappa}\frac{\mathscr{E}^2}{r^4}\right) r^2\, dr, \tag{1.7a}$$

$$= 1 - 2(\mu + \varepsilon)/r + \mathscr{E}^2/r^2, \tag{1.7b}$$

$$\gamma(r) = \int_0^r \frac{e^{\alpha}}{r}\left(1 - e^{-\alpha} - \frac{\mathscr{E}^2}{r^2} + \kappa r^2 M_1^1\right) dr, \tag{1.8a}$$

$$= -\alpha(r) + \kappa\int_0^r r e^{\alpha}(\rho + M_1^1)\, dr. \tag{1.8b}$$

The functions $\mathscr{E}$, μ and ε are defined by

$$\mathscr{E}(r) = 4\pi\int_0^r \sigma r^2 e^{\alpha/2}\, dr, \tag{1.9}$$

$$\mu(r) = 4\pi\int_0^r \rho r^2\, dr, \tag{1.10}$$

$$\varepsilon(r) = 4\pi\int_0^r r\sigma\mathscr{E} e^{\alpha/2}\, dr, \tag{1.11}$$

where $\rho(r)$ $(=-M_4^4)$ is the proper matter density. The function $\mathscr{E}(r)$ is, obviously, the total charge inside a sphere of radius r. We shall write

$$e = \mathscr{E}(a) \tag{1.12}$$

for the total charge of the distribution. For later use we note that (1.9) implies

$$\mathscr{E}' = 4\pi\sigma r^2 e^{\alpha/2}, \tag{1.13}$$

where the prime denotes differentiation with respect to r.

As was pointed out in paper A, the functions $\alpha(r)$ and $\gamma(r)$ in the above solution are functionals of the basic functions $\rho(r)$, $\sigma(r)$ and $M_1^1(r)$. These basic functions are assumed continuous throughout the interior of the distribution and, of course, zero outside it. It is further assumed that $\rho(r)$ is positive for all r and that M_1^1 satisfies the junction condition

$$M_1^1 = [C] = 0 \quad \text{across } S. \tag{1.14}$$

Making these assumptions, the above solutions for $\alpha(r)$ and $\gamma(r)$ are well behaved everywhere and they satisfy the conditions for elementary flatness $\alpha(0)=0$, $\gamma(0)=0$, and the boundary conditions

$$\alpha=[C], \qquad \gamma=[C]=\gamma' \qquad \text{across } S. \tag{1.15}$$

The remaining components M_2^2 and M_3^3 $(=M_2^2)$ of the matter tensor are given in terms of ρ, σ and M_1^1 by the equation

$$M_2^2=M_3^3=\tfrac{1}{2}r(M_1^1)'+(1+\tfrac{1}{4}r\gamma')M_1^1+\tfrac{1}{4}r(\rho\gamma'-2\sigma r^{-2}\mathscr{E}e^{\alpha/2}). \tag{1.16}$$

The only non-vanishing component, F_{14}, of the electromagnetic tensor F_{ij} is given by

$$F_{14}=\mathscr{E}r^{-2}e^{(\alpha+\gamma)/2} \tag{1.17}$$

and the only non-vanishing components of the electromagnetic energy tensor E_j^i are given by

$$E_1^1=-E_2^2=-E_3^3=E_4^4=-\kappa^{-1}\mathscr{E}^2r^{-4}. \tag{1.18}$$

It is evident that F_{ij} and E_j^i are well behaved for all values of r and are continuous across S: $r=a$.

In the exterior of the distribution, the solution (1.7) and (1.8) reduce to (paper A)

$$e^{-\alpha}=1-\frac{2m}{r}+\frac{e^2}{r^2}, \qquad e^{\gamma}=\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)e^{C}, \tag{1.19}$$

where C denotes the constant

$$C=\kappa\int_0^a re^{\alpha}(\rho+M_1^1)\,\mathrm{d}r, \tag{1.20}$$

and m, the gravitational mass of the distribution, the constant

$$m=\mu(a)+\varepsilon(a). \tag{1.21}$$

It was shown in A that $\varepsilon(a)$ is the mass equivalence of the electromagnetic energy of the distribution. The solution (1.19) is, of course, the (exterior) Nordström solution.

As was pointed out in A the functions $\alpha(r)$ and $\gamma(r)$ are not *explicitly* defined by (1.7) and (1.8) in the interior of the distribution; for from equation (1.9) the function $\mathscr{E}(r)$ in terms of which $\alpha(r)$ and $\gamma(r)$ are defined is itself dependent on $\alpha(r)$. This, however, does not prevent us from obtaining *explicit* solutions and proving some interesting results as was done in paper A for charged dust distributions. In any case, it is evident from the solutions (1.7) and (1.8) that these solutions do become explicit if, instead of $\sigma(r)$, we regard $\mathscr{E}(r)$ as one of the basic functions mentioned earlier; the charge density $\sigma(r)$ can then be read from equation (1.13).

The aim of the present paper is to obtain further explicit solutions from the general (implicit) solution (1.7) and (1.8). These solutions represent the complete field of (I) charged perfect fluid spheres, and (II) charged distributions with no radial stress.

Perfect fluidity is normally defined (Synge 1966) by the requirement that the three eigenvalues, corresponding to the three space-like eigenvectors, of the energy tensor shall all be equal everywhere. In the present case we have before us *three* energy tensors, namely, the matter energy tensor M_j^i, the electromagnetic energy tensor E_j^i and the total energy tensor $T_j^i=M_j^i+E_j^i$. The possibility, therefore, arises of defining perfect fluidity by the requirement that the three eigenvalues of either (Ia) T_j^i, or (Ib)

M^i_j, or (Ic) E^i_j shall all be equal everywhere (see concluding remark on s§ 2). The possibility (Ic) can be dismissed because, in view of equation (1.18), it is implied that $\mathscr{E}(r)=0$ for all r and, therefore, $\sigma(r)=0$ for all r. The possibilities (Ia) and (Ib) are considered in § 2 where explicit solutions are given in each case. Both sets of solutions reduce to the Schwarzschild (interior) solution when the charge density is set equal to zero.

In an analogous way we define systems (II), that is systems with no radial stress, by the requirement that the radial component of either (IIa) the total energy tensor T^i_j, or (IIb) the matter tensor M^i_j shall be zero everywhere. A number of explicit solutions, some of them extremely simple, are given in each case in § 3. When the charge density is set equal to zero all these solutions (with $\rho \neq 0$) reduce, inside the distributions, to the new Schwarzschild solution obtained by the author (Florides 1974). Accordingly, we make the conjecture that these solutions represent the field inside a *charged* Einstein cluster.

2. Charged perfect fluid spheres

In this section we apply the general solution given by equations (1.7) and (1.8) to find the complete field of a static charged perfect fluid sphere. As explained in the introduction we can define perfect fluidity for charged distributions by the requirement that the three eigenvalues, corresponding to the three space-like eigenvectors, of either (Ia) the total energy tensor $T^i_j=M^i_j+E^i_j$, or (Ib) the matter tensor M^i_j, shall be equal everywhere. We shall now consider these two cases separately.

2.1. Case (Ia)

The requirement that the above mentioned three eigenvalues of $T^i_j=M^i_j+E^i_j$ shall be equal implies that

$$T^1_1=T^2_2=T^3_3. \tag{2.1}$$

Using equations (1.2) and (1.18) this gives

$$M^2_2=M^3_3=M^1_1-(2/\kappa)(\mathscr{E}^2/r^4). \tag{2.2}$$

We shall write

$$M^1_1=p(r), \tag{2.3}$$

where, in view of the boundary condition (1.14),

$$p(a)=0. \tag{2.4}$$

Using equations (2.2) and (2.3) in (1.16) we get

$$\tfrac{1}{2}rp'+\tfrac{1}{4}r\gamma'(p+\rho)+(2/\kappa)(\mathscr{E}^2/r^4)-\tfrac{1}{2}(\sigma\mathscr{E}/r)e^{\alpha/2}=0. \tag{2.5}$$

But, from equation (1.8b),

$$\gamma'=-\alpha'+re^{\alpha}(p+\rho). \tag{2.6}$$

Substituting this in equation (2.5) and making use of (1.13) we get

$$Z'-\tfrac{1}{2}\alpha'Z+\tfrac{1}{2}\kappa re^{\alpha}Z^2-(\rho+\mathscr{E}^2/\kappa r^4)'=0, \tag{2.7}$$

where we have written

$$Z = p + \rho. \tag{2.8}$$

Since $\rho(r)$ and $\sigma(r)$ are assumed to be given, the functions $\alpha(r)$, and consequently $\mathscr{E}(r)$, are known. Thus equation (2.7) is an equation for $Z(r)$ and, hence, for $p(r)$. If we can find Z from (2.7) then the function $\gamma(r)$ can be found from equation (1.8b) thus determining the complete field of the charge distribution under consideration.

Equation (2.7) is recognised as the Riccati equation. As is well known, in order to integrate this equation we first have to find a particular integral for it; the equation can then be integrated by quadratures. We have, however, been unable to find such a particular integral for equation (2.7) for arbitrary $\rho(r)$ and $\sigma(r)$. It is worth pointing out that equation (2.7) is formally identical to the corresponding equation for uncharged perfect fluid spheres (Synge 1966). This is, of course, not surprising in view of equation (2.1). Even in the neutral case a particular integral of the equation corresponding to (2.7) is elusive.

To make the integration of equation (2.7) possible we are forced to introduce some simplifying assumptions on the form of $\rho(r)$ and $\sigma(r)$ (or $\mathscr{E}(r)$). We shall look for assumptions which, in addition to simplifying the mathematics, will yield a solution of the Einstein–Maxwell equations which is closely analogous to the neutral (interior) Schwarzschild solution and to which our solution will reduce when the charge density is set equal to zero. Looking at equation (2.7) the obvious simplifying assumption is

$$\rho + \mathscr{E}^2/\kappa r^4 = \text{constant} = c, \text{ say}. \tag{2.9}$$

Cooperstock and De La Cruz (1978) were the first authors to use this assumption. (Their work was confined to charged *dust* spheres and to charged perfect fluid spheres with *zero matter density*; perfect fluidity they defined as in (Ib) above.) Since the total charge of the system is assumed finite (and equal to e) the physical requirement $\rho \geqslant 0$ is satisfied if the constant c in (2.9) satisfies

$$c \geqslant e^2/\kappa a^4. \tag{2.10}$$

The consequence of the assumption (2.9) is twofold. It simplifies (2.7) to

$$Z' - \tfrac{1}{2}\alpha' Z + \tfrac{1}{2}\kappa r e^{\alpha} Z^2 = 0, \tag{2.11}$$

and, by equation (1.7a), it results in a particularly simple form for $\alpha(r)$, namely,

$$e^{-\alpha} = 1 - r^2/R^2, \tag{2.12}$$

where we have written

$$1/R^2 = \kappa c/3. \tag{2.13}$$

We note that signature considerations imply that

$$a < R \quad \text{or} \quad \kappa c a^2 < 3. \tag{2.14}$$

From equation (2.12) we get

$$\alpha' = 2rR^{-2}e^{\alpha} \tag{2.15}$$

which enables us to write (2.11) in the form

$$Z^{-1}(1 - \tfrac{1}{2}\kappa R^2 Z)^{-1}\, \mathrm{d}Z = \tfrac{1}{2}\mathrm{d}\alpha. \tag{2.16}$$

This integrates to

$$Z \equiv p + \rho = (2R^{-2}/\kappa)[1 + B(1 - r^2/R^2)^{1/2}]^{-1}, \tag{2.17}$$

where B is the constant of integration. To determine B we use the conditions $p(a) = 0$, and (from equations (1.12) and (2.9))

$$\rho(a) = c - e^2/\kappa a^4 \equiv \rho_a. \tag{2.18}$$

We find that

$$B = (2 - \kappa R^2 \rho_a)(\kappa R^2 \rho_a)^{-1}(1 - a^2/R^2)^{-1/2}. \tag{2.19}$$

It is worth pointing out that equation (2.17) is formally identical to the corresponding expression for $(p + \rho)$ in the neutral (interior Schwarzschild) case (Synge 1966). The two expressions differ only in the value of the constant† B. When $\sigma(r)$ is set equal to zero, equation (2.17) reduces to the corresponding expression in the Schwarzschild solution (with constant matter density c).

We proceed now to evaluate the function $\gamma(r)$ using equation (1.8b), namely

$$\gamma(r) = -\alpha(r) + \kappa J(r) \tag{2.20a}$$

where we have written

$$J(r) = \int_0^r re^{\alpha}(p + \rho)\,\mathrm{d}r = \int_0^r re^{\alpha} Z\,\mathrm{d}r. \tag{2.20b}$$

The easiest way to evaluate this integral is to use equation (2.15) to express re^{α} in terms of α'. We get

$$J(r) = \int_0^r \tfrac{1}{2}R^2 Z\alpha'\,\mathrm{d}r = R^2 \int_{Z(0)}^{Z(r)} (1 - \tfrac{1}{2}\kappa R^2 Z)^{-1}\,\mathrm{d}Z,$$

having used equation (2.16) in the last step. Integrating and using equations (2.17) and (2.12) we get

$$\kappa J(r) = \alpha(r) + 2\log\{(1 + B)^{-1}[1 + B(1 - r^2/R^2)^{1/2}]\}.$$

Hence, by equation (2.20a), we get

$$e^{\gamma} = \{[1 + B(1 - r^2/R^2)^{1/2}]/1 + B\}^2. \tag{2.21}$$

Summarising, the solution inside the distribution under consideration is given by

$$\mathrm{d}s^2 = (1 - r^2/R^2)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - \{[1 + B(1 - r^2/R^2)^{1/2}]/1 + B\}^2\,\mathrm{d}t^2, \tag{2.22}$$

where R and B are constants given by equations (2.13) and (2.19), respectively. The similarity between this solution and the neutral interior Schwarzschild solution (with constant matter density c) is as remarkable as it is unexpected. The only difference between the two solutions lies in the different values of the constant B in the two cases. When we set $\sigma(r) = 0$ in (2.22) we get $e = 0$, $\rho = c = \rho_a$, $B = -\tfrac{1}{3}(1 - a^2/R^2)^{-1}$, and the solution (2.22) reduces to the interior Schwarzschild solution with constant matter density c.

† In the interior Schwarzschild case with constant matter density c, $(p + c)$ can be expressed in the form (2.17) where, now, R is given as in (2.13) and B by $B = (2 - \kappa R^2 c)(\kappa R^2 c)^{-1}(1 - a^2/R^2)^{-1/2} = -\tfrac{1}{3}(1 - a^2/R^2)^{-1/2}$.

The solution outside the distribution is, of course, the (exterior) Nordström solution given by (1.19). The total mass m is defined by equation (1.21) which, in view of equations (1.12), (1.13) and (2.9) gives

$$m=\tfrac{1}{3}4\pi a^3c+\tfrac{1}{2}(e^2/a), \tag{2.23}$$

e being the total charge of the distribution. For the constant C in (1.20) we get, using equations (2.17), (2.15) and (2.16),

$$e^C=[1+B(1-a^2/R^2)^{1/2}]^2[(1+B)(1-a^2/R^2)^{1/2}]^{-2} \tag{2.24}$$

As was pointed out in paper A the constant C can be reduced to zero by rescaling the time coordinate. It is worth emphasising again that the solution (2.22), as indeed all the solutions in this paper, matches smoothly to the (exterior) Nordström solution.

Finally, we remark that the functions $\rho(r)$ and $\mathscr{E}(r)$ associated with the solution (2.22) are, apart from equation (2.9) which they have to satisfy, quite arbitrary.

2.2. *Case (Ib)*

We recall that in this case it is required that the three eigenvalues, corresponding to the three space-like eigenvectors, of the matter tensor M^i_j shall be equal everywhere. This implies

$$M^1_1=M^2_2=M^3_3=p(r),\ \text{say}. \tag{2.25}$$

Since, by equation (1.14), $M^1_1(a)=0$ we must have

$$p(a)=0. \tag{2.26}$$

The steps leading to the solution (see (2.43) below) for this case are formally the same as for case (Ia) and many of the details in the present case will, accordingly, be omitted. The same steps leading to (2.7) in case (Ia) now lead to

$$Z'-\tfrac{1}{2}\alpha'Z+\tfrac{1}{2}\kappa re^{\alpha}Z^2-4\mathscr{E}^2/\kappa r^5-(\rho+\mathscr{E}^2/\kappa r^4)'=0, \tag{2.27}$$

where

$$Z=p+\rho. \tag{2.28}$$

Equation (2.27) differs from the corresponding equation (2.7) of case (Ia) only by the extra factor $-4\mathscr{E}^2\kappa^{-1}r^{-5}$. To simplify equation (2.27) we impose, as in (Ia), the condition

$$\rho+\mathscr{E}^2/\kappa r^4=\text{constant}=c,\ \text{say}. \tag{2.29}$$

Then, as in (Ia),

$$e^{-\alpha}=1-r^2/R^2, \tag{2.30}$$

with

$$1/R^2=\tfrac{1}{3}\kappa c. \tag{2.31}$$

This simplifies (2.27) to

$$Z'=\tfrac{1}{4}(2Z-\kappa R^2Z^2)\alpha'+4\mathscr{E}^2/\kappa r^5. \tag{2.32}$$

This equation is still of the Riccati type and a particular integral of it is still elusive. A possible way of integrating (2.32) is to make it separable by choosing the term

$4\mathscr{E}^2/(\kappa r^5)$ proportional to α'. It is convenient to write

$$4\mathscr{E}^2/\kappa r^5 = \tfrac{1}{4}A\alpha' \tag{2.33}$$

where $A/4$ is the constant of proportionality. The constant A can be evaluated by the requirement that $e = \mathscr{E}(a)$ giving

$$A = e^2R^2(1-a^2/R^2)/\pi a^6. \tag{2.34}$$

By equations (1.13) and (2.30) the assumption (2.33) is tantamount to choosing the charge density

$$\sigma(r) = \pm(A/2\kappa R^2)^{1/2}(3+r^2e^\alpha/R^2) = \pm(A/2\kappa R^2)^{1/2}(3-2r^2/R^2)(1-r^2/R^2)^{-1}. \tag{2.35}$$

We return now to equation (2.32) which, in view of equation (2.33), separates into

$$(A+2Z-\kappa R^2Z^2)^{-1}\,\mathrm{d}Z = \tfrac{1}{4}\,\mathrm{d}\alpha. \tag{2.36}$$

This can readily be integrated to give

$$Z \equiv p+\rho = \frac{1}{R^2}\left(\frac{(n+1)-B(n-1)(1-r^2/R^2)^{n/2}}{1+B(1-r^2/R^2)^{n/2}}\right) \tag{2.37}$$

where we have written

$$n = (1+\kappa R^2A)^{1/2} \tag{2.38}$$

and B is the constant of integration. To determine B we use the boundary conditions $p(a)=0$ and (from equations (1.12) and (2.29))

$$\rho(a) = c - e^2/\kappa a^4 \equiv \rho_a, \tag{2.39}$$

say. We find that

$$B = (n+1-\kappa R^2\rho_a)(n-1+\kappa R^2\rho_a)^{-1}(1-a^2/R^2)^{-n/2}. \tag{2.40}$$

As a check on these calculations we note that if we put $\sigma(r)=0$, then $e=0$, $n=1$, $B=-\tfrac{1}{3}(1-a^2/R^2)^{-1/2}$ and equation (2.37) reduces to the corresponding equation in the interior Schwarzschild solution.

We now proceed to determine the function $\gamma(r)$. The method is the same as in case (Ia). We write equation (1.8*b*) in the form (2.20*a*), namely,

$$\gamma(r) = -\alpha(r) + \kappa J(r), \tag{2.41}$$

where

$$J(r) = \int_0^r re^\alpha Z\,\mathrm{d}r.$$

Since, as in case (Ia), $\alpha' = 2rR^{-2}e^\alpha$ this integral becomes

$$J(r) = \tfrac{1}{2}R^2\int_0^r Z\alpha'\,\mathrm{d}r = 2R^2\int_{Z(0)}^{Z(r)} Z(A+2Z-\kappa R^2Z^2)^{-1}\,\mathrm{d}Z,$$

making use of equation (2.36) in the last step. Integrating and using equation (2.37) we get

$$\kappa J(r) = 2\log\left(\frac{1+B(1-r^2/R^2)^{n/2}}{1+B}\right) - \tfrac{1}{2}(n+1)\log\left(1-\frac{r^2}{R^2}\right).$$

Substituting this in equation (2.41) and making use of equation (2.30) we get

$$e^{\gamma}=\left(\frac{1+B(1-r^2/R^2)^{n/2}}{1+B}\right)^2\left(1-\frac{r^2}{R^2}\right)^{-(n-1)/2}. \tag{2.42}$$

Summarising, the solution inside the distribution under consideration is given by

$$ds^2=\left(1-\frac{r^2}{R^2}\right)^{-1}dr^2+r^2\,d\Omega^2-\left(\frac{1+B(1-r^2/R^2)^{n/2}}{1+B}\right)^2\left(1-\frac{r^2}{R^2}\right)^{-(n-1)/2}dt^2, \tag{2.43}$$

where R, B and n are constants given by equations (2.31), (2.40) and (2.38), respectively. The charge density $\sigma(r)$ is given by equation (2.35) and, in view of the equations (2.29), (2.30) and (2.33), the matter density of the distribution is given by

$$\rho(r)=c-\tfrac{1}{8}AR^{-2}r^2(1-r^2/R^2)^{-1}, \tag{2.44}$$

the constant A being given by equation (2.34). Thus unlike case (Ia) in which $\sigma(r)$ and $\rho(r)$ are arbitrary functions constrained only by the condition (2.9), in case (Ib) they have the definite values given by equations (2.35) and (2.44). We note that $\rho(r)$ is a monotonically decreasing function of r and that it remains positive throughout the distribution provided

$$c>e^2/\kappa a^4. \tag{2.45}$$

Signature considerations of the solution (2.43) imply

$$R^2>a^2, \quad \text{or} \quad 3>\kappa ca^2. \tag{2.46}$$

These last two conditions are the same as in case (Ia). The last condition ensures that the solutions in both cases are real.

A comparison between the solutions (2.22) of case (Ia) and (2.43) of case (Ib) indicates, at first sight, that the former is a particular case of the latter when n, in (2.43), is set equal to unity. This, however, is not the case. Indeed, with $n=1$ equations (2.38) and (2.34) show that $e^2=0$. It follows immediately then that the solution (2.43) reduces to the (neutral) interior Schwarzschild solution with constant matter density c.

The solution outside the distribution is, of course, the Nordström solution (1.19). The total mass m can be found in the same way as in case (Ia) and it is, in fact, exactly the same as the mass in case (Ia) given by equation (2.23). For the constant C in (1.20) we get, using the same method as in case (Ia),

$$e^{C}=\left(\frac{1+B(1-a^2/R^2)^{n/2}}{1+B}\right)^2. \tag{2.47}$$

As we pointed out above, both solutions (2.22) and (2.43), reduce to the Schwarzschild interior solutions with $\rho=c$ when the charge density is set equal to zero. Indeed, both solutions seem to be the simplest generalisation of the Schwarzschild solution for charged fluid spheres.

The only restrictions on the solutions are the inequalities (see equations (2.10), (2.14) and (2.45), (2.46))

$$e^2/a^2<\kappa ca^2<3. \tag{2.48}$$

These follow from the requirement that the matter density must be positive and the signature must be +2 throughout the fluid. In terms of a, e and the total mass m

(given by (2.23)) the above inequalities take the form

$$4e^2 < 6ma < 3(a^2+e^2). \tag{2.49}$$

A look at equations (2.22) and (2.43) shows that, in order to ensure that the solutions are well behaved everywhere we must impose the additional condition

$$1+B \neq 0, \tag{2.50}$$

where B is the constant given by (2.19) for case (Ia) and by (2.40) for case (Ib). The same condition (2.50) ensures that the pressures $p(r)$ in both cases are finite everywhere. Although (2.50) is essentially a condition on the relative magnitudes of a, e and m, we have been unable to express it in any simple and useful form.

Finally a word about the definition of perfect fluidity for charged distributions. In addition to the usual definition, (Ib), in terms of the equality of the three eigenvalues of the matter tensor M^i_j, we gave the definition (Ia) in terms of the equality of the eigenvalues of the total energy tensor $T^i_j = M^i_j + E^i_j$. This was not an idle exercise. Indeed, if gravitation and electromagnetism were fused together in a unified theory it is definition (Ia), rather than (Ib), that should be adopted.

3. Static spherically symmetric charged systems with no radial stress

Following the pattern of § 2 we shall define charged distributions with no radial stress by the three alternative requirements:

(IIa) the radial component, T^1_1, of the total energy tensor $T^i_j = M^i_j + E^i_j$ is zero,

(IIb) the radial component, M^1_1, of the matter tensor M^i_j is zero, and

(IIc) the radial component, E^1_1, of the electromagnetic energy tensor E^i_j is zero.

The definition (IIc) can be ruled out immediately since, by equation (1.18), it implies the complete absence of charge. We consider the first two definitions in turn.

3.1. Case (IIa)

By definition (IIa) we have $T^1_1 = M^1_1 + E^1_1 = 0$. In view of equation (1.18) this gives

$$M^1_1 = \mathscr{E}^2/\kappa r^4. \tag{3.1}$$

Since $M^1_1 = 0$ outside the distribution, the boundary condition (1.14) is no longer satisfied. A look at equation (1.8*a*), however, shows that, in view of (3.1), the component M^1_1 drops out of the definition of $\gamma(r)$. Hence the boundary condition (1.15) for $\gamma(r)$ remains satisfied.

Substituting (3.1) in (1.7*a*) and (1.8*a*) we get, respectively,

$$e^{-\alpha(r)} = 1 - 2\mathscr{M}(r)/r \tag{3.2}$$

and

$$\gamma(r) = \int_0^r 2r^{-2} e^{\alpha} \mathscr{M}(r)\, dr, \tag{3.3}$$

where we have written

$$\mathscr{M}(r) = 4\pi \int_0^r \left(\rho + \frac{\mathscr{E}^2}{\kappa r^4}\right) r^2\, dr. \tag{3.4}$$

Since ρ, σ (or $\mathscr{E}$) are assumed known, equations (3.2) and (3.3) give the complete field of the distribution under consideration. With M_1^1 also known (by equation (3.1)), equation (1.16) becomes the defining equation for M_2^2, M_3^3 $(=M_2^2)$. Using equations (3.1)–(3.4) and equation (1.13) in equation (1.16) we get

$$M_2^2 = M_3^3 = \tfrac{1}{2}r^{-1}(1-2\mathcal{M}(r)/r)^{-1}(\rho+\mathscr{E}^2/\kappa r^4)\mathcal{M}(r) - \mathscr{E}^2/\kappa r^4. \tag{3.5}$$

Equations (3.1)–(3.5) represent the complete field of the most general charge distribution with $T_1^1 = 0$. Outside the distribution the solution is, as always, the (exterior) Nordström solution (1.19). The total mass m is defined by (1.21) and for the solution just obtained it is given by

$$m = \tfrac{1}{2}e^2/a + \mathcal{M}(a). \tag{3.6}$$

The constant C in equation (1.20) is now given by

$$C = \kappa \int_0^a re^{\alpha}\left(\rho + \frac{\mathscr{E}^2}{\kappa r^4}\right) \mathrm{d}r.$$

A look at the general solution (3.1)–(3.5) shows that it can be considerably simplified if, as in § 2, we impose the condition

$$\rho + \mathscr{E}^2/\kappa r^4 = c = \text{constant}. \tag{3.7}$$

Then (3.4) gives, inside the distribution,

$$\mathcal{M}(r) = \tfrac{1}{2}r^3/R^2, \qquad (R^{-2} = \kappa c/3), \tag{3.8}$$

and the solution (3.2), (3.3) becomes

$$\mathrm{d}s^2 = (1-r^2/R^2)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - (1-r^2/R^2)^{-1/2}\,\mathrm{d}t^2 \tag{3.9}$$

with

$$M_2^2 = M_3^3 = \tfrac{2}{3}\pi c^2 r^2(1-r^2/R^2)^{-1} - \mathscr{E}^2/\kappa r^4. \tag{3.10}$$

Many more explicit solutions can be obtained from the general solution (3.2), (3.3) by giving suitable values to ρ and σ (or $\mathscr{E}$). For example the choice

$$\rho = \text{constant}, \ \sigma = \sigma_0 e^{\alpha(r)/2}, \qquad (\sigma_0 = \text{constant}),$$

or the choice

$$\rho = 0, \qquad \mathscr{E}^2 = A(n+1)r^{n+2} \tag{3.11}$$

where A and n are positive constants, with $n \geqslant 4$, give explicit solutions quite readily. We shall consider the latter choice for its simplicity. Using (3.11) in (3.2)–(3.4) we get the following solution inside the distribution:

$$\mathrm{d}s^2 = (1-Ar^n)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - (1-Ar^n)^{-1/n}\,\mathrm{d}t^2. \tag{3.12}$$

For the matter tensor (3.5) now gives

$$M_3^2 = M_3^3 = \tfrac{1}{4}A(n+1)r^{n-2}(5Ar^n - 4)(1-Ar^n)^{-1}, \qquad M_4^4 = 0.$$

M_1^1 is, of course, given by (3.1) and (3.11). The physical meaning of the constant A can be found from the fact that $\mathscr{E}(a) = e$ = total charge of the distribution. Equation (3.11) then gives

$$A = e^2/(n+1)a^{n+2}. \tag{3.13}$$

By equation (1.13) the choice (3.11) is tantamount to choosing

$$\sigma = \pm\kappa^{-1}[(n+1)A]^{1/2}(n+2)r^{(n-4)/2}(1-Ar^n)^{1/2}. \tag{3.14}$$

It is because of this formula that we imposed the condition $n \geqslant 4$; the charge density σ would have otherwise been singular at $r=0$.

The continuation of the solution (3.12) outside the distribution is given, as always, by the Nordström solution (1.19). The gravitational mass m of the system is now given by

$$m = e^2(n+2)/2a(n+1), \tag{3.15}$$

and the constant C in (1.20) by

$$e^C = (1-Aa^n)^{-(n-1)/n}.$$

It is seen from equation (3.15) that the gravitational mass of the system is purely of electromagnetic origin. That such a possibility is admitted by general relativity was pointed out by the author in 1962. This possibility was also demonstrated more recently by Cooperstock and De La Cruz (1978).

3.2. *The neutral case*

Let us put $\sigma = 0 = \mathscr{E}$ in the general solution (3.1)–(3.5). The solution reduces to

$$ds^2 = \left(1 - \frac{2\mu(r)}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2 - \exp\left(\int_0^r \frac{2\mu(r)\, dr}{r^2(1-2\mu(r)/r)}\right) dt^2, \tag{3.16}$$

with the (matter) energy tensor

$$M_1^1 = 0, \qquad M_2^2 = M_3^3 = \frac{\mu(r)\rho(r)}{2r(1-2\mu(r)/r)}, \qquad M_4^4 = -\rho(r), \tag{3.17}$$

where

$$\mu(r) = 4\pi \int_0^r \rho(r)\, dr. \tag{3.18}$$

This solution is precisely the 'new interior Schwarzschild solution' obtained by the author in 1974.

Again put $\sigma = 0 = \mathscr{E}$ in the particular solution (3.9), (3.10). We get

$$ds^2 = (1-r^2/R^2)^{-1}\, dr^2 + r^2\, d\Omega^2 - (1-r^2/R^2)^{-1/2}\, dt^2, \tag{3.19}$$

with the (matter) energy tensor

$$M_1^1 = 0, \qquad M_2^2 = M_3^3 = \tfrac{2}{3}\pi c^2 r^2 (1-r^2/R^2)^{-1}, \qquad M_4^4 = -c. \tag{3.20}$$

We have thus recovered the special 'new interior Schwarzschild solution' (Florides 1974) (it can, also, be obtained from the solution (3.16) by putting $\rho(r) = c =$ constant).

3.3. *Case (IIb)*

In this case we have, by the definition given at the beginning of this section,

$$M_1^1 = 0. \tag{3.21a}$$

Substituting this in (1.7*a*) and (1.8*a*) we get

$$e^{-\alpha(r)} = 1 - 2\mathcal{M}(r)/r, \tag{3.21b}$$

$$\gamma(r) = \int_0^r \frac{e^\alpha}{r}\left(\frac{2\mathcal{M}(r)}{r} - \frac{\mathscr{E}^2}{r^2}\right) dr, \tag{3.21c}$$

where, as in the previous case, we have written

$$\mathcal{M}(r) = 4\pi \int_0^r \left(\rho + \frac{\mathscr{E}^2}{\kappa r^4}\right) r^2 \, dr. \tag{3.22}$$

With ρ and σ (or $\mathscr{E}$) assumed known, equations (3.21) give the complete field of the distribution under consideration. They also determine the remaining components M_2^2, M_3^3 of the matter tensor M_j^i via equation (1.13). They are given by

$$\begin{aligned} M_2^2 = M_3^3 &= \tfrac{1}{4} r(\rho\gamma' - 2\sigma r^{-2}\mathscr{E} e^{\alpha/2}) \\ &= \tfrac{1}{4} e^{\alpha/2}[\rho e^{\alpha/2}(2\mathcal{M}(r)/r - \mathscr{E}^2/r^2) - 2\sigma\mathscr{E}/r], \end{aligned} \tag{3.23}$$

where we have used (3.21*b*) in the last step. Equations (3.21)–(3.23) represent the complete field of the most general charge distribution with $M_1^1 = 0$. Outside the distribution the field is the (exterior) Nordström solution (1.19) with m given, as in the previous case, by (3.6) and the constant C by

$$C = \kappa \int_0^a \rho r e^\alpha \, dr. \tag{3.24}$$

Of the many special (explicit) solutions that can be obtained from the general solution (3.21) we confine ourselves to the following two solutions.

Solution 1

We take, as on previous occasions,

$$\rho + \mathscr{E}^2/\kappa r^4 = c = \text{constant}, \qquad \text{with } (c \geqslant e^2/a^4). \tag{3.25}$$

Equation (3.22) then gives

$$2\mathcal{M}(r) = r^3/R^2, \qquad (1/R^2 = \kappa c/3), \tag{3.26}$$

so that equation (3.21*a*) becomes

$$e^{-\alpha(r)} = 1 - r^2/R^2. \tag{3.27}$$

Using the last two equations in (3.21*b*) we get, after an integration,

$$\gamma(r) = \tfrac{1}{2}\alpha(r) - \int_0^r \frac{e^\alpha \mathscr{E}^2}{r^3} \, dr. \tag{3.28}$$

To make further progress we now choose

$$\mathscr{E}^2 = B(n+1) r^{n+3} e^{-\alpha(r)}, \tag{3.29}$$

where B and n are positive constants with $n \geqslant 3$. In view of equations (3.25) and (1.13) this choice is equivalent to taking

$$\rho = c - \kappa^{-1} B(n+1) r^{n-1}(1 - r^2/R^2), \tag{3.30}$$

and

$$\sigma = \pm\kappa^{-1}[(n+1)B]^{1/2}r^{(n-3)/2}[(n+3)-(n+5)r^2/R^2]. \qquad (3.31)$$

This last equation explains the restriction $n \geqslant 3$ on the otherwise arbitrary constant n. From the requirement $\mathscr{E}(a) = e$, the total charge of the distribution, the constant B in (3.29) is given by

$$B = (n+3)^{-1}e^2a^{-(n+3)}(1-a^2/R^2)^{-1}.$$

With the choice (3.29) equation (3.28) now gives

$$\gamma = \tfrac{1}{2}\alpha - Br^{n+1}$$

so that the field inside this particular charge distribution is given by

$$ds^2 = (1-r^2/R^2)^{-1}\,dr^2 + r^2\,d\Omega^2 - e^{-Br^{n+1}}(1-r^2/R^2)^{-1/2}\,dt^2. \qquad (3.32)$$

The continuation of this solution outside the distribution is given by the Nordström solution with the constants m and C given by

$$m = \tfrac{4}{3}\pi c a^3 + \tfrac{1}{2}e^2/a, \qquad C = -\tfrac{3}{2}\log(1-a^2/R^2) - Ba^{n+1}.$$

Solution 2

We choose ρ and $\mathscr{E}$ as in (3.11), namely

$$\rho = 0, \qquad \mathscr{E}^2 = A(n+1)r^{n+2}, \qquad (3.33)$$

where A and n are arbitrary constants with $n \geqslant 4$. Then from the general solution (3.21) we get the simple special solution

$$ds^2 = (1-Ar^n)^{-1}\,dr^2 + r^2\,d\Omega^2 - (1-Ar^n)\,dt^2 \qquad (3.34)$$

inside the distribution under consideration. The constant A is given as in (3.13) and the charge density as in (3.14). But for the only two non-vanishing components of the matter tensor we now have, from equation (3.23),

$$M_2^2 = M_3^3 = -\tfrac{1}{2}r^{-1}\sigma\mathscr{E}e^{\alpha/2} = -\tfrac{1}{2}\kappa^{-1}A(n+1)(n+2)r^{n-2}. \qquad (3.35)$$

For the field outside the distribution we have the Nordström solution with m and C given by

$$m = e^2(n+2)/2a(n+1), \qquad \text{(as in (3.15))}, \qquad (3.36)$$

and $C = 0$.

The electromagnetic origin of the gravitational mass is, again, obvious from (3.36).

Finally, let us point out that if we put $\sigma = 0 = \mathscr{E}$ in the general solution (3.21) we recover the 'new interior Schwarzschild solution' (3.16). Thus both general solutions for cases (IIa) and (IIb) reduce to the (neutral) 'new interior Schwarzschild solution' on setting the charge density σ equal to zero. Now, in the 1974 paper, in which this new interior solution was presented, the author proved the following result. The 'new interior Schwarzschild solution' describes the field inside that Einstein (neutral) cluster (Einstein 1939, Florides and Jones 1970) whose matter density $\tilde{\rho}$ is related to the matter density ρ of the solution (3.16) by the equation

$$\tilde{\rho}(r) = \rho(r)(1-3\mu(r)/r)(1-2\mu(r)/r)^{-1} \qquad (3.37)$$

where $\mu(r)$ is given by (3.18). Since the 'new interior Schwarzschild solution' (3.16) can be derived from the two general solutions (3.2), (3.3), and (3.21) for systems (II), it is natural to ask whether these two solutions represent the field inside an Einstein *charged* cluster whose matter and charge densities are related in some specific way to the matter and charge densities of our two solutions. We believe the answer to this question to be affirmative. For the moment, however, this has to be left as a conjecture.

References

Cooperstock F I and De La Cruz V 1978 *Gen. Rel. and Grav.* **9** 835
Einstein A 1939 *Ann. Math.* **40** 922
Florides P S 1962 *Proc. Camb. Phil. Soc.* **58** 102
—— 1974 *Proc. R. Soc.* A**337** 592
—— 1977 *Nuovo Cimento* **42A** 343
Florides P S and Jones R L 1970 *Nuovo Cimento* **69** 41
Synge J L 1966 *Relativity: the general theory* (Amsterdam: North-Holland)